JOURNAL OF DIFFERENTIAL EQUATIONS 30, 287–295 (1978)

On a Comparison Theorem

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Received July 29, 1977

This paper gives a generalization of the Sturm comparison theorem for differential equations (p): $y'' = p(t)\,y$, (q): $y'' = q(t)\,y$ under the assumption that the function $p - q$ changes its sign exactly once on $[a, b]$ or $\int_t^b p - q$, $\int_a^t p - q$ maintain the sign on $[a, b]$. The results are used for investigating the distributions of zeros of solutions and the derivative of solutions of (p), (q).

1. INTRODUCTION

A number of papers of fairly recent origin have appeared [2, 3, 4] dealing with some generalizations of the classical Sturm comparison theorem for differential equations (p): $y'' = p(t)\,y$, (q): $y'' = q(t)\,y$ under the assumption that the function $p - q$ changes its sign exactly once on the interval considered. The results obtained are then utilized in deriving inequalities between the values of the first derivatives of the first and third kinds of basic central dispersions of the differential equations (p), (q) [3, 4]. This paper presents one more generalization of the Sturm comparison theorem with inequalities derived even between the values of the first derivatives of the first, second, third, and fourth kinds of basic central dispersions of the differential equations (p), (q).

2. DEFINITIONS, NOTATION, AND PRELIMINARY RESULTS

Let

$$(\mathrm{q}) \quad y'' = q(t)\,y$$

be a differential equation for which q is C^0 on an interval $i = (A, B)$, $-\infty \leqslant A < B \leqslant \infty$, and let j be a closed subinterval $[a, b] \subset i$.

Let y be a nontrivial solution of (q) vanishing at t_0 ($\in i$) and let $\phi_{(\mathrm{q})}(t_0)$ be the first zero of y lying on the right of t_0. The function $\phi_{(\mathrm{q})}$ is called the basic central dispersion of the first kind of (q) (briefly the first kind of dispersion of (q)). Any such dispersion satisfies

$$\phi_{(\mathrm{q})}(t) > t, \qquad \phi_{(\mathrm{q})} \in C^3, \qquad \phi'_{(\mathrm{q})}(t) > 0$$

on its domain of definition (which may also be an empty set).

0022-0396/78/0303-0287$02.00/0

Let $q(t) < 0$ on i and y_1, y_2 be nontrivial solutions of (q), $y_1(t_0) = 0$, $y_2'(t_0) = 0$. If $\psi_{(q)}(t_0)$ is the first zero of y_2' on the right of t_0, then $\psi_{(q)}$ is called the second kind of basic central dispersion of (q) (briefly the second kind of dispersion of (q)). If $\chi_{(q)}(t_0)$ $(\omega_{(q)}(t_0))$ is the first zero of y_1' (y_2) lying on the right of t_0, then $\chi_{(q)}$ $(\omega_{(q)})$ is called the third (fourth) kind of basic central dispersion of (q) (briefly the third (fourth) kind of dispersion of (q)). If $\gamma_{(q)}$ is the kth dispersion of (q), $k = 2, 3, 4$, then

$$\gamma_{(q)}(t) > t, \qquad \gamma_{(q)} \in C^1, \qquad \gamma'_{(q)}(t) > 0$$

on its domain of definition (which may also be an empty set) (see [1]).

Instead of $\phi_{(q)}(t_0)$, $\chi_{(q)}(t_0)$, and $\omega_{(q)}(t_0)$, i.e., the values of the first, third, and fourth kinds of dispersions of (q) in t_0 we often meet with terms of the first conjugate point of the point t_0, the first focal point of the point t_0 and the second focal point of the point t_0, respectively, (see [3, 4, 6]). Since we investigate here the distribution of zeros of the derivative of solutions of (q) as well (described by the basic central dispersion of the second kind of (q)), we apply the definitions, notations, and results from [1] only.

If $\phi_{(q)}$, $\psi_{(q)}$, $\chi_{(q)}$, and $\omega_{(q)}$ are the first, second, third, and fourth kinds of dispersions of (q) defined in t_0 $(\in i)$ and y_1, y_2 are nontrivial solutions of (q), $y_1(t_0) = 0$, $y_2'(t_0) = 0$, then

$$\phi'_{(q)}(t_0) = \frac{y_1'^2(t_0)}{y_1'^2[\phi_{(q)}(t_0)]}, \tag{1}$$

$$\psi'_{(q)}(t_0) = \frac{q(t_0)}{q[\psi_{(q)}(t_0)]} \frac{y_2^2(t_0)}{y_2^2[\psi_{(q)}(t_0)]}, \tag{2}$$

$$\chi'_{(q)}(t_0) = -\frac{1}{q[\chi_{(q)}(t_0)]} \frac{y_1'^2(t_0)}{y_1^2[\chi_{(q)}(t_0)]}, \tag{3}$$

$$\omega'_{(q)}(t_0) = -q(t_0) \frac{y_2^2(t_0)}{y_2'^2[\omega_{(q)}(t_0)]} \tag{4}$$

(cf. [1, pp. 120, 121]).

In [5] it is proved that the functions $\psi_{(q)}$, $\chi_{(q)}$, $\omega_{(q)}$ may be defined for every (q) so that, in case of $q(t) < 0$ on i, they coincide with the second, third, and fourth kinds of dispersions of (q), respectively. Besides $\chi_{(q)}$ need not be continuous; the function $\omega_{(q)}$ is continuous including its first derivative and can be proved (in analogy to [1, pp. 120, 121]) that the formula (4) holds.

For a C^2-function $q(t) \neq 0$ defined on i, define $q^*(t) = q(t) - \frac{1}{2}q''(t)/q(t) + \frac{3}{4}(q'(t)/q(t))^2$. We shall need:

LEMMA ([1, p. 9]). *There is one-to-one correspondence of solutions $y(t)$ of (q) and $y^*(t)$ of (q*) given by*

$$y^*(t) = y'(t) \mid q(t)\mid^{-1/2}. \tag{5}$$

Remark. For $q < 0$, the distribution of zeros of the derivative of a solution of (q) can be studied via the first kind of dispersion of (q*).

We say that the function f defined on j ($= [a, b]$) changes its sign exactly once on j if there exists a point c, $c \in j$, such that $f(t) \geqslant 0$ ($f(t) \leqslant 0$) for $t \in [a, c)$ and $f(t) \leqslant 0$ ($f(t) \geqslant 0$) for $t \in [c, b]$.

The trivial solutions are excluded from our considerations.

3. Results

THEOREM 1. *Let u, v be solutions of* (p), (q), *respectively. Let the next conditions be valid:*

(i) $q(a) > p(a)$, $q(b) < p(b)$, $q(b) \leqslant 0$ *and the function* $q(t) - p(t)$ *changes its sign exactly once on* j,

(ii) $u(t) \geqslant 0$, $v(t) \geqslant 0$ *for* $t \in j$,

(iii) $u(a) = v(a)$, $u'(a) = v'(a)$,

(iv) $u(b) \cos \beta - u'(b) \sin \beta = 0$, $v(b) \cos \beta - v'(b) \sin \beta = 0$.

Then

$$u(b) \leqslant v(b), \qquad | u'(b)| \leqslant | v'(b)|,$$

and the equality $u(b) = v(b)$ ($| u'(b)| = | v'(b)|$) *occurs only in case of* $\sin \beta = 0$ ($\cos \beta = 0$).

Remark. Theorem 1 has been proved in [3] by assuming that $q(t) < 0$, $p(t) < 0$ for $t \in j$. For a special case, where $\sin \beta = 0$, this theorem has been proved in [4].

Proof. (a) Let $q(a) > p(a)$. We prove the existence of such a right neighborhood $(a, a + \epsilon)$ of the point a where

$$u''(t) < v''(t), \qquad t \in (a, a + \epsilon), \tag{6}$$

holds true. We devide the proof into five parts according to the relative signs of $p(a)$, $q(a)$.

(1) $0 > q(a) > p(a)$. Then $\lim_{t \to a+0} u''(t)/v''(t) = p(a)/q(a) > 1$ and from $v'' = qv < 0$ we have $u''(t) < v''(t)$ for $t \in (a, a + \epsilon)$.

(2) $q(a) > p(a) > 0$. Then $\lim_{t \to a+0} u''(t)/v''(t) = p(a)/q(a) < 1$ and from $v'' = qv > 0$ we have $u''(t) < v''(t)$ for $t \in (a, a + \epsilon)$.

(3) $q(a) > 0 > p(a)$. From $v'' = qv > 0$, $u'' = pu < 0$ we have $u''(t) < v''(t)$ for $t \in (a, a + \epsilon)$.

(4) $0 = q(a) > p(a)$. Then $\lim_{t \to a+0} v''(t)/u''(t) = q(a)/p(a) = 0$ and from $u'' = pu < 0$ we have $u''(t) < v''(t)$ for $t \in (a, a + \epsilon)$.

(5) $q(a) > p(a) = 0$. Then $\lim_{t\to a+0} u''(t)/v''(t) = p(a)/q(a) = 0$ and from $v'' = qv > 0$ we have $u''(t) < v''(t)$ for $t \in (a, a + \epsilon)$.

We now have proved (6) in all cases and from (iii) we infer that in the interval also

$$u'(t) < v'(t), \qquad u(t) < v(t)$$

are true.

(b) We now define the function w on j, $w(t) = u(t)\,v'(t) - u'(t)\,v(t)$ (in analogy with [3, 4]). From (iii) and (iv) we get $w(a) = w(b) = 0$. From $w'(t) = (q(t) - p(t))\,u(t)\,v(t)$, $q(a) > p(a)$, (ii) and (iii) it follows that $w(t) > 0$ for all t lying in a right neighborhood of the point a. Since $w(a) = w(b) = 0$ and $w'(t) = 0$ iff $q(t) = p(t)$, Rolle's theorem implies $w(t) > 0$ on (a, b). If $u(t) < v(t)$ is not true on all of (a, b), it fails for the first time at a point ξ, $\xi \in (a, b)$: $u(\xi) = v(\xi)$. From

$$(u(t) - u(\xi))/(t - \xi) > (v(t) - v(\xi))/(t - \xi), \qquad t \in (a, \xi),$$

we get $u'(\xi) \geqslant v'(\xi)$ and therefore $w(\xi) = (v'(\xi) - u'(\xi))\,u(\xi) \leqslant 0$ which contradicts the inequality $w(t) > 0$, $t \in (a, b)$ proved before.

(c) Let $p(b) > q(b) \leqslant 0$. In analogy with part (a) of the proof we can now prove the existence of such a left neighborhood $(b - \epsilon, b)$ of the point b where

$$u''(t) > v''(t), \qquad t \in (b - \epsilon, b), \tag{7}$$

holds true. We divide the proof into four parts according to the relative signs of $p(b)$, $q(b)$.

(1) $0 > p(b) > q(b)$. Then $0 > p(t) > q(t)$, $u''(t)/v''(t) = (p(t)\,u(t))/(q(t)\,v(t)) < 1$ and from $v'' = qv < 0$ we have $u''(t) > v''(t)$ for $t \in (b - \epsilon, b)$.

(2) $q(b) < 0 < p(b)$. Then $q(t) < 0 < p(t)$ and from $u'' = pu > 0$, $v'' = qv < 0$ we have $u''(t) > v''(t)$ for $t \in (b - \epsilon, b)$.

(3) $q(b) = 0 < p(b)$. Then $\lim_{t\to b-0} v''(t)/u''(t) = \lim_{t\to b-0} q(t)/p(t) \cdot \lim_{t\to b-0} v(t)/u(t) = 0$ and from $u'' = pu > 0$ we have $u''(t) > v''(t)$ for $t \in (b - \epsilon, b)$.

(4) $q(b) < p(b) = 0$. Then $\lim_{t\to b-0} u''(t)/v''(t) = \lim_{t\to b-0} p(t)/q(t) \cdot \lim_{t\to b-0} u(t)/v(t) = 0$ and from $v'' = qv < 0$ we have $u''(t) > v''(t)$ for $t \in (b - \epsilon, b)$.

(d) Integrating (7) gives

$$u'(b) - u'(t) > v'(b) - v'(t), \qquad t \in (b - \epsilon, b). \tag{8}$$

In analogy with [3] we divide the next part of the proof into three parts.

(1) If $\cos\beta = 0$, then $u'(b) = v'(b) = 0$, $u'(t) < v'(t)$ for $t \in (b - \epsilon, b)$. Consequently

$$u(b) = u(t) + \int_t^b u'(s)\,ds < v(t) + \int_t^b v'(s)\,ds = v(b).$$

(2) If $\sin\beta = 0$, then $u(b) = v(b) = 0$ and from (8) we get

$$u'(b) - v'(b) > (v(t) - u(t))/(b - t) > 0, \qquad t \in (b - \epsilon, b).$$

Since $u'(b) < 0$, $v'(b) < 0$, it holds that $|u'(b)| < |v'(b)|$.

(3) Let $\sin\beta\cos\beta \neq 0$. If $u(b) = v(b)$, then analogous to case (2) we can prove $|u'(b)| \neq |v'(b)|$. Since $u'(b) = \cot\beta u(b)$ and $v'(b) = \cot\beta v(b)$, we obtain $u'(b) = v'(b)$, a contradiction. Therefore $u(b) < v(b)$ and $|u'(b)| < |v'(b)|$.

This completes the proof of Theorem 1.

Corollary 1. *If the conditions of Theorem* 1 *hold for*

(a) $u(a) = v(a) = u(b) = v(b) = 0$,

(b) $u(a) = v(a) = 0$, $u'(b) = v'(b) = 0$; $p(t) < 0$, $q(t) < 0$ *for* $t \in i$,

(c) $u'(a) = v'(a) = 0$, $u(b) = v(b) = 0$, $0 \geqslant q(a)$

then, in case (a) $\phi'_{(\mathrm{p})}(a) > \phi'_{(\mathrm{q})}(a)$, *in case* (b) $\chi'_{(\mathrm{p})}(a) > \chi'_{(\mathrm{q})}(a)$ *and in case* (c) $\omega'_{(\mathrm{p})}(a) > \omega'_{(\mathrm{q})}(a)$.

Proof. (a) If $u(a) = v(a) = u(b) = v(b) = 0$, then $\phi_{(\mathrm{q})}(a) = \phi_{(\mathrm{p})}(a) = b$. Therefore the formulas $\phi'_{(\mathrm{p})}(a) = u'^2(a)/u'^2(b)$, $\phi'_{(\mathrm{q})}(a) = v'^2(a)/v'^2(b)$ which follow from (1) and $u'^2(b) < v'^2(b)$, proved in Theorem 1, lead us to $\phi'_{(\mathrm{p})}(a) > \phi'_{(\mathrm{q})}(a)$.

(b) If $u(a) = v(a) = 0$, $u'(b) = v'(b) = 0$, then $\chi_{(\mathrm{q})}(a) = \chi_{(\mathrm{p})}(a) = b$. Therefore the formulas $\chi'_{(\mathrm{p})}(a) = -u'^2(a)/(p(b)\,u^2(b))$, $\chi'_{(\mathrm{q})}(a) = -v'^2(a)/(q(b)\,v^2(b))$ which follow from (3), $q(b) < p(b) < 0$ and the inequality $u^2(b) < v^2(b)$, proved in Theorem 1, lead to $\chi'_{(\mathrm{p})}(a) > \chi'_{(\mathrm{q})}(a)$.

(c) According to [5], $\omega_{(\mathrm{q})}$ and $\omega_{(\mathrm{p})}$ have a continuous derivative and the formula (4) holds. If $u'(a) = v'(a) = 0$, $u(b) = v(b) = 0$, then $\omega_{(\mathrm{q})}(a) = \omega_{(\mathrm{p})}(a) = b$. Consequently $\omega'_{(\mathrm{p})}(a) = -p(a)\,u^2(a)/u'^2(b)$, $\omega'_{(\mathrm{q})}(a) = -q(a)\,v^2(a)/v'^2(b)$. According to Theorem 1 it is $u'^2(b) < v'^2(b)$. According to the assumption it holds $0 \geqslant q(a) > p(a)$ and therefore the inequalities $\omega'_{(\mathrm{p})}(a) > 0$, $\omega'_{(\mathrm{q})}(a)/\omega'_{(\mathrm{p})}(a) = (q(a)\,u'^2(b))/(p(a)\,v'^2(b)) < 1$ result in $\omega'_{(\mathrm{p})}(a) > \omega'_{(\mathrm{q})}(a)$.

Remark. Case (a) of Corollary 1 has been proved in [4] under the additional assumption $q(t) < 0$, $p(t) < 0$ on i. Case (b) of Corollary 1 has been proved in [3]. Let us also mention the impossibility of deleting the assumption $p(t) < 0$, $q(t) < 0$ for $t \in i$ in case (b) for its removal would cause that $\chi'_{(\mathrm{p})}(a)$ or $\chi'_{(\mathrm{q})}(a)$ need not generally exist (see [5]).

COROLLARY 2. *Let u, v be the solutions of* (p), (q), *respectively*, $p(t) < 0$, $q(t) < 0$ *for* $t \in i$, $p(t), q(t)$ *the* C^2*-functions defined on* i *such that* $u(a)(-p(a))^{1/2} = v(a)(-q(a))^{1/2}$, $u'(a) = v'(a) = u'(b) = v'(b) = 0$, $u'(t) > 0$, $v'(t) > 0$ *for* $t \in (a, b)$. *Let* $q^*(a) > p^*(a)$, $q^*(b) < p^*(b)$, $q^*(b) \leqslant 0$ *and let the function* $q^*(t) - p^*(t)$ *changes its sign exactly once on* j. *Then*

$$u(b)(-p(b))^{1/2} < v(b)(-q(b))^{1/2}$$

and

$$\psi'_{(p)}(a) > \psi'_{(q)}(a).$$

Proof. According to our lemma the functions $\bar{u}(t) = u'(t)(-p(t))^{-1/2}$, $\bar{v}(t) = v'(t)(-q(t))^{-1/2}$ are the solutions of (p*), (q*), respectively, and next it holds that $\bar{u}(a) = \bar{v}(a) = \bar{u}(b) = \bar{v}(b) = 0$, $\bar{u}'(a) = \bar{v}'(a) > 0$. From Theorem 1 where instead of p, q and u, v we consider p^*, q^* and $\bar{u}$, $\bar{v}$, we obtain $0 > \bar{u}'(b) > \bar{v}'(b)$. Since $\bar{u}'(b) = -u(b)(-p(b))^{1/2}$, $\bar{v}'(b) = -v(b)(-q(b))^{1/2}$, it follows that $0 < u(b)(-p(b))^{1/2} < v(b)(-q(b))^{1/2}$ and $0 > p(b)\, u^2(b) > q(b)\, v^2(b)$. Now following (2) we can write ($\psi_{(q)}(a) = \psi_{(p)}(a) = b$)

$$\psi'_{(p)}(a) = (p(a)\, u^2(a))(p(b)\, u^2(b))^{-1}, \qquad \psi'_{(q)}(a) = (q(a)\, v^2(a))(q(b)\, v^2(b))^{-1}.$$

According to the assumption it holds $p(a)\, u^2(a) = q(a)\, v^2(a)$ and therefore $\psi'_{(p)}(a) > \psi'_{(q)}(a)$.

In Theorem 1 an important role has been played by the assumption saying that the function $p(t) - q(t)$ changes its sign exactly once on j. This assumption may be weakened in certain cases as can be seen from the next theorems.

THEOREM 2. *Let u, v be the solutions of* (p), (q), *respectively. Let the next conditions be valid*:

(i) $u(a) = v(a)$, $u'(a) = v'(a)$, $u'(b) = v'(b) = 0$,

(ii) $u(t) > 0$, $v(t) > 0$, $u'(t) > 0$, $v'(t) > 0$ *for* $t \in (a, b)$,

(iii) $q(a) > p(a)$, $\int_t^b (p(s) - q(s))\, ds > 0$ *for* $t \in (a, b)$.

Then

$$u(t) < v(t) \qquad \textit{for} \quad t \in (a, b],$$

$$u'(t) < v'(t) \qquad \textit{for} \quad t \in (a, b).$$

Proof. Completely analogous to part (a) of the proof of Theorem 1 we prove the validity of inequality (6) and thus also of the inequalities $u'(t) < v'(t)$, $u(t) < v(t)$ for $t \in (a, a + \epsilon)$. For the function w, $w(t) = u(t)\, v'(t) - u'(t)\, v(t)$ defined on j we have $w(a) = w(b) = 0$, $w'(t) = (q(t) - p(t))\, u(t)\, v(t)$. By integrating the last equality from t ($\in j$) to b and using the integration by parts we can write

$$\begin{aligned} w(b) - w(t) &= \int_t^b w'(s)\,ds = \int_t^b (q(s) - p(s))\,u(s)\,v(s)\,ds \\ &= u(t)\,v(t) \int_t^b (q(s) - p(s))\,ds \\ &\quad + \int_t^b \left[(u'(s)\,v(s) + u(s)\,v'(s)) \int_s^b (q(z) - p(z))\,dz\right] ds, \end{aligned}$$

which shows that

$$\begin{aligned} w(t) = &- u(t)\,v(t) \int_t^b (q(s) - p(s))\,ds \\ &- \int_t^b \left[(u'(s)\,v(s) + u(s)\,v'(s)) \int_s^b (q(z) - p(z))\,dz\right] ds \end{aligned}$$

holds. According to the assumption we have $\int_t^b (q(s) - p(s))\,ds < 0$ for $t \in (a, b)$ and therefore with respect to assumption (ii) we have $w(t) > 0$ for $t \in (a, b)$. The inequality $u(t) < v(t)$ for $t \in (a, b]$ can be derived from the last inequality analogously to part (b) of the proof of Theorem 1. If $u'(\xi) = v'(\xi)$, $\xi \in (a, b)$, then $w(\xi) = (u(\xi) - v(\xi))\,u'(\xi) < 0$, which is a contradiction. Therefore $u'(t) < v'(t)$ for $t \in (a, b)$.

Corollary 3. *If the conditions of Theorem 2 hold and $p(t) < 0$, $q(t) < 0$ for $t \in i$, $q(b) < p(b)$, then*

$$\chi'_{(\mathrm{p})}(a) > \chi'_{(\mathrm{q})}(a).$$

Proof. The method of proof is analogous to that of part (b) of the proof in Corollary 1.

Theorem 3. *Let u, v be the solutions of* (p), (q), *respectively, and let the following conditions be true*:

(i) $u(a) = v(a)$, $u'(a) = v'(a) = 0$, $u(b) = v(b) = 0$,

(ii) $u(t) > 0$, $v(t) > 0$, $u'(t) < 0$, $v'(t) < 0$ *for* $t \in (a, b)$,

(iii) $q(a) > p(a)$, $\int_a^t (p(s) - q(s))\,ds < 0$ *for* $t \in (a, b)$.

Then

$$u(t) < v(t) \qquad \text{for} \quad t \in (a, b)$$

and

$$u'(b) \geqslant v'(b).$$

Proof. Similarly to part (a) of the proof of Theorem 1 we will prove

$$u(t) < v(t), \qquad u'(t) < v'(t) \qquad \text{for} \quad t \in (a, a + \epsilon).$$

For the function w, $w(t) = u(t)\,v'(t) - u'(t)\,v(t)$ defined on j, we have $w(a) = w(b) = 0$, $w'(t) = (q(t) - p(t))\,u(t)\,v(t)$. Integrating the last equality from a to t ($\in j$) and using the integration by parts we have

$$w(t) - w(a) = \int_a^t w'(s)\,ds = \int_a^t (q(s) - p(s))\,u(s)\,v(s)\,ds$$
$$= u(t)\,v(t) \int_a^t (q(s) - p(s))\,ds$$
$$- \int_a^t \left[(u'(s)\,v(s) + u(s)\,v'(s)) \int_a^s (q(z) - p(z))\,dz\right] ds$$

so that

$$w(t) = u(t)\,v(t) \int_a^t (q(s) - p(s))\,ds$$
$$- \int_a^t \left[(u'(s)\,v(s) + u(s)\,v'(s)) \int_a^t (q(z) - p(z))\,dz\right] ds$$

holds true. And thus with respect to (ii) and (iii) we get $w(t) > 0$ for $t \in (a, b)$. It then follows from part (b) of the proof of Theorem 1 that $u(t) < v(t)$ for $t \in (a, b)$. The inequality $u'(b) \geqslant v'(b)$ will be obtained passing to the limit $(t \to b)$ in the inequality

$$(u(t) - u(b))/(t - b) > (v(t) - v(b))/(t - b), \qquad t \in (a, b).$$

COROLLARY 4. *If the conditions of Theorem* 3 *hold and* $0 \geqslant q(a)$ ($> p(a)$), *then*

$$\omega'_{(\mathrm{p})}(a) > \omega'_{(\mathrm{q})}(a).$$

Proof. The method of proof is the same as that of part (c) of Corollary 1.

COROLLARY 5. *Let* u, v *be the solutions of* (p), (q), *respectively, and let the following conditions hold true*:

(i) $u(a) = v(a) = 0$, $u'(a) = v'(a)$, $u(b) = v(b) = 0$, $u(t) > 0$, $v(t) > 0$ *for* $t \in (a, b)$,

(ii) *there exists a number* c, $c \in (a, b)$ *such that* $u'(t) > 0$, $v'(t) > 0$ *for* $t \in [a, c)$, $u'(t) < 0$, $v'(t) < 0$ *for* $t \in (c, b]$,

(iii) $q(a) > p(a)$, $\int_t^c (p(s) - q(s))\,ds > 0$ *for* $t \in (a, c)$,

$$\int_c^t (p(s) - q(s))\,ds < 0 \qquad \textit{for} \quad t \in (c, b).$$

Then

$$u(t) < v(t) \qquad \textit{for } t \in (a, b)$$

and

$$\phi'_{(p)}(a) \geqslant \phi'_{(q)}(a).$$

If $q(b) < p(b) \leqslant 0$, *it even holds that* $\phi'_{(p)}(a) > \phi'_{(q)}(a)$.

Proof. Let us put $w(t) = u(t)\,v'(t) - u'(t)\,v(t)$ for $t \in j$. Then $w(a) = w(c) = w(b) = 0$ and $w(t) > 0$ for $t \in (a, c) \cup (c, d)$. According to Theorem 2 we have $u(c) < v(c)$. Consequently $u(t) < v(t)$ for $t \in (a, b)$ and it holds that $0 > u'(b) \geqslant v'(b)$. The inequality $\phi'_{(p)}(a) \geqslant \phi'_{(q)}(a)$ immediately follows from the formulas $\phi'_{(p)}(a) = u'^2(a)\,u'^{-2}(b)$, $\phi'_{(q)}(a) = v'^2(a)\,v'^{-2}(b)$. If $q(b) < p(b) \leqslant 0$, then the inequality $\phi'_{(p)}(a) > \phi'_{(q)}(a)$ follows from the inequality $u'(b) > v'(b)$ which could be proved in the same way as that of part (d) of the proof in Theorem 1.

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